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D-branes on noncommutative orbifolds

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Abstract: We study tachyon condensation on noncommutative toric orbifolds with a $\mathbb{Z}_2$ discrete group and explore the various kinds of brane bound states arising in the case of irrational values of the B-field. We show that $\mathbb{Z}_2$ symmetry of the orbifolds incorporates naturally anti-branes in the spectrum and leads to equivalent results as those obtained by starting from an original pair of D-$\overline{\mathrm{D}}$ system on quantum torii. A specific analysis is deserved to the irrational representation of NC orbifolds and to the unstable bound states generated by the condensation.

Keywords: D-branes, Tachyon Condensation, Non-Commutative Geometry.

Contents

1. Introduction

Recently a great interest has been given to the study of tachyon condensation using non-commutative (NC) geometry [1]. Starting from non-BPS D-branes of string theory and turning on an antisymmetric NS-NS B-field, one gets a condensation of tachyon fields on the world volume of the lower dimensional branes [2]. The key step in the derivation of this result is based on the computation of the vacuum energy configurations after neglecting the kinetic part of the string field effective action in front of the potential term [3]. The energy of the soliton is shown to be proportional to the trace of projectors on the ground states of the non commutative algebra Hilbert space [4].

Using GMS construction [5], NC tachyon condensation has been first studied in [6] and especially electric fluxons obtained from original non-BPS D-branes on the non compact Moyal spaces in the presence of a constant B-field and second in [7], see also [8, 9], in terms of D0-branes by starting from a D2-brane on a NC two torus. In [10], we have extended the above mentioned results to more general cases; in particular to higher dimensional torii where we have shown the existence of general unstable bound states decaying into D0-D2 ones and suggested an effective potential to describe these brane states. In the study performed in [10], it has remained however a discussion regarding the D-$\overline{\mathrm{D}}$ systems on non-commutative compact manifolds. Such systems are interesting in the study of unstable D brane systems in type-II string theory as they go beyond the theory of non BPS D-branesrecovered by restricting to couplings invariant under $(-)^{F_L}$ [11] and were already considered from the view point of complex tachyon condensation on the quantum two torus. But here we shall reconsider this analysis differently using the discrete symmetries of the NC brane world volumes. The present study may be then viewed as a continuation of the analysis we initiated in [10] by first completing our previous results to the case of non-BPS branes on non-commutative toric orbifolds. We also study the D0-D2 bound states of [7] and explore the nature of branes generated by the tachyon condensation in NC $\mathbb{T}_\theta^2/\mathbb{Z}_2$. Then we analyze the spectrum of brane bound states one gets generally when studying tachyon condensation of D-branes wrapped on higher dimensional orbifolds.

The aim of this work is to start from an original non-BPS $D2l$-brane and study the NC solitons in $\mathbb{Z}_2$ toric orbifolds for both rational and irrational NC θ_i's parameters. We first study tachyon in $\mathbb{T}_\theta^2/\mathbb{Z}_2$ and consider the extension to higher dimensional compact orbifolds. Then, we derive the various kinds of solitons and explore all types of bound states one gets after the condensation. Finally we study the D-$\overline{\mathrm{D}}$ brane systems from different views; once by starting from a pair of D-$\overline{\mathrm{D}}$ brane on quantum torii and second by considering only a D-brane on orbifolds.

The paper is organized as follows. In section 2, we describe two different representations of the NC $\mathbb{Z}_2$ orbifolds according to whether the NC parameter θ is rational or not. In section 3, we analyze the two cases of non BPS branes on the NC orbifold torus constructed in section 2 depending on whether θ's are rational or not. In section 4, we study the D-$\overline{\mathrm{D}}$ brane systems either by using standard analysis based on starting from an original pair of D-$\overline{\mathrm{D}}$ branes or by using the $\mathbb{Z}_2$ symmetry of the orbifolds. In section 5, we study the D-$\overline{\mathrm{D}}$ system and in section 6 we give a summary and discussion. The last section is devoted for conclusion.

Note in passing that works treating the special examples of the $\mathbb{Z}_2$ orbifolds of the two dimensional Moyal plane and quantum torus concerning noncommutative solitons were considered in [12]–[15]. The present study goes beyond this analysis; it completes and generalize partial results on branes on NC orbifolds discussed in the previous mentionned papers. In particular, we explicitly exhibit the role of discrete torsions by using the $\mathbb{Z}_2^{(\mathrm{qunt})}$ quantum symmetry associated to the geometric $\mathbb{Z}_2$ orbifold symmetry. Moreover, we give representations of the NC structure at both regular and singular points and analyse the structure of fractional branes.

2. Solitons in non-commutative orbifolds

Here we study briefly the construction of non-commutative orbifolds $\mathbb{T}_\theta^{2l}/\mathbb{Z}_2$ by starting from the usual realisation of non-commutative $\mathbb{T}_\theta^{2l}$ and imposing $\mathbb{Z}_2$ symmetry. Then we study its rational and irrational representations and give the projectors on the ground state configurations which will be used later in the building of solitons in $\mathbb{T}_\theta^{2l}/\mathbb{Z}_2$.

To start recall that the $\mathbb{T}_\theta^{2l}$ we will be considering is roughly speaking given by the product of l non-commutative two dimensional torii $\mathbb{T}_{\theta_i}^2$ generated by a system of l unitary operator pairs (U_i, V_i) satisfying the algebra [10]

$$U_i V_i = e^{-i2\pi\theta_i} V_i U_i\,, \qquad i = 1, \ldots, l \tag{2.1}$$

$$U_i V_j = V_j U_i\,; \qquad i \neq j\,, \tag{2.2}$$

for which we will shall refer from now on as $\mathfrak{A}_\theta$, the non-commutative algebra of functionals on the torus $\mathbb{T}^{2l}_\theta$. Note that because of eqs. (2.1), (2.2), this non-commutative algebra $\mathfrak{A}_\theta$ may be also defined as the tensor product of l factors $\mathfrak{A}_{\theta_i}$ as $\mathfrak{A}_\theta = \otimes_{i=1}^{l}\mathfrak{A}_{\theta_i}$; where each $\mathfrak{A}_{\theta_i}$ factor is associated with the non-commutative torus $\mathbb{T}^2_{\theta_i}$; and the corresponding (U_i, V_i) pairs are realized as the exponential of the non-commutative coordinates (x^{2i-1}, x^{2i}) of $\mathbb{T}^2_{\theta_i}$. For later use we prefer to denote the coordinates pairs of the $\mathbb{T}^2_{\theta_i}$ non-commutative torii by the capital letters (X^{2i-1}, X^{2i}) while those of the commutative ones by small Latin letters. Thus we have

$$\begin{aligned} U_i &= \mathrm{e}^{\frac{i2\pi}{R_{2i-1}}X^{2i-1}} ; && (2.3)\\ V_i &= \mathrm{e}^{\frac{i2\pi}{R_{2i}}X^{2i}} , \qquad i = 1, \ldots, l && (2.4)\end{aligned}$$

where the R_j's are the one cycles radii of the $2l$ dimensional torus. Note also that the adjoint operator pairs (U_i^+, V_i^+), which may read as

$$\begin{aligned} U_i^+ &= e^{-\frac{i2\pi}{R_{2i-1}}X^{2i-1}} ; && (2.5)\\ V_i^+ &= e^{-\frac{i2\pi}{R_{2i}}X^{2i}} , \qquad i = 1, \ldots, l , && (2.6)\end{aligned}$$

satisfy the following equivalent identities.

$$\begin{aligned} U_i^+ V_i^+ &= \mathrm{e}^{-i2\pi\theta_i} V_i^+ U_i^+ , && (2.7)\\ U_i^+ V_j^+ &= V_j^+ U_i^+ ; \qquad i \neq j . && (2.8)\end{aligned}$$

Now we turn to define the orbifold $\mathbb{T}^{2l}_\theta/\mathbb{Z}_2$ as the quotient of $\mathbb{T}^{2l}_\theta$ by $\mathbb{Z}_2$. In fact, an element Ω of $\mathbb{Z}_2$ identifies the local coordinates $\{x^I\}$ $(x^I \equiv x^I + 2\pi R^I)$ with $\{-x^I\}$, $(1 \leq I \leq 2l)$. While the (U_i, V_i) and (U_i^+, V_i^+) pairs defining the non-commutative orbifold are related as:

$$\begin{aligned} \Omega U_i \Omega &= U_i^+ , \qquad \Omega^2 = I_{id} \\ \Omega V_i \Omega &= V_i^+ , \qquad 1 \leq i \leq l . && (2.9)\end{aligned}$$

To solve these equations, we will distinguish two cases according to whether the θ_i's are rational or not and so two different representations for the orbifold. We shall use both of these representations; that's why we propose to first describe them briefly on the simple case of the non-commutative $\mathbb{T}^2_\theta/\mathbb{Z}_2$ orbifold; then we extend our result to the general situation. To begin note that natural solutions of eqs. (2.9) are given by setting the U_i's an V_i's as:

$$U_i = \begin{pmatrix} A_i & E_i \\ C_i & D_i \end{pmatrix} , \qquad V_i = \begin{pmatrix} B_i & F_i \\ G_i & H_i \end{pmatrix} , \qquad (2.10)$$

where A_i, B_i, C_i, D_i, E_i, F_i, G_i and H_i are operators to be determined later. Putting this change in eqs. (2.9), one gets:

$$\begin{aligned} D_i &= A_i^+ , \qquad C_i = E_i^+ , \\ H_i &= B_i^+ , \qquad G_i = F_i^+ . && (2.11)\end{aligned}$$

At this level we would like to emphasize that the above constraints are inherent to the $\mathbb{Z}_2$ orbifold in the sense that all A_i, B_i, C_i, D_i, E_i, F_i, G_i and H_i matrix operators should have the same dimension. This will constitute a key property in distinguishing rational and irrational $\mathbb{Z}_2$ orbifolds and will play an important role when we discuss the solitons problem. Moreover, and as far as the study of solitons on non-commutative spaces is concerned, one can further impose $E_i = C_i = 0$. This special choice will be justified later on when we build the projectors Π_i and $\mathcal{P}_{N_i+M_i\theta_i}$; see eqs. (2.15) and (2.21), on the vacuum field configurations. For convenience we shall also require that $F_i = G_i = 0$. This representation is a particular one which will help us to simplify the analysis; it is reducible into irreducible factors (A_i, B_i) and their hermitian conjugates.

Rational representations. This representation corresponds to rational values of $\theta_i = q_i/p_i$, where p_i and q_i are mutually coprime integers. The A_i and B_i operators are given by the following finite $p_i \times p_i$ matrices

$$A_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega_i & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \omega_i^{p_i-1} \end{bmatrix}; \qquad B_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.12}$$

where $\omega_i = \mathrm{e}^{i2\pi q_i/p_i}$. Note in passing that $A_i^{p_i}$ and $B_i^{p_i}$ act as the $p_i \times p_i$ identity operator $I_{p_i\times p_i}$, and so is $U_i^{p_i}$ and $V_i^{p_i}$. Therefore any element e_i of the non-commutative algebra $\mathfrak{A}_{\theta_i}$ associated to $\mathbb{T}^2_\theta/\mathbb{Z}_2$ has a finite expansion

$$e_i = \sum_{n,m=0}^{p_i-1} (e_i)_{nm} U_i^n V_i^m \,. \tag{2.13}$$

In the matrix representation presented above, the U_i generators are given by diagonal matrices; a feature which allows to build the usual k_i-th rank Π_i projectors $(\Pi_i)_{k_i} = \mathrm{diag}(1, 1, \ldots, 1, 0, \ldots, 0; 1, 1, \ldots, 1, 0, \ldots, 0)$, as a series of the U_i's as shown here below:

$$(\Pi_i)_{k_i} = \sum_{n=0}^{p_i-1} (e_i)_{n0} U_i^n \,. \tag{2.14}$$

A direct check shows that the $(e_i)_{n0}$ coefficients are given by $(e_i)_{n0} = \frac{1}{p_i}\frac{1-\omega_i^{-nk_i}}{1-\omega_i^{-n}}$. Note that according to eq. (2.10), the Π_i projectors may be rewritten in terms of the A_i's and their hermitian $A_i^+ \equiv \overline{A}_i$ conjugates as

$$(\Pi_i)_{k_i} = \sum_{n=0}^{p_i-1} (e_i)_{n0} \begin{pmatrix} (A_i)^n & 0 \\ 0 & (\overline{A}_i)^n \end{pmatrix}, \tag{2.15}$$

or equivalently

$$\Pi_i = \pi_i \oplus \overline{\pi}_i \,,$$

where

$$(\pi_i)_{k_i} = \sum_{n=0}^{p_i-1} (e_i)_{n0} A_i^n \,, \tag{2.16}$$

$$(\overline{\pi}_i)_{k_i} = \sum_{n=0}^{p_i-1} (\overline{e}_i)_{(p_i-n)0} \overline{A}_i^n \,. \tag{2.17}$$

Since the trace on $\mathfrak{A}_{\theta_i}$ is given by $\mathrm{Tr}(\Pi_i)_{k_i} = (e_i)_{00} + (\overline{e}_i)_{p_i 0} = (k_i + \overline{k}_i) = 2k_i$, then the range of k_i which is bounded as $0 \leq k_i \leq p_i$ will be interpreted as giving the number of D0 and $\overline{\text{D0}}$ branes one obtains from the study of the condensation of a non BPS D2-brane on the NC $\mathbb{T}_\theta^2/\mathbb{Z}_2$. Note moreover that the Π_i projector we built satisfy naturally

$$\Omega \Pi_i \Omega = \Pi_i \,, \qquad \Omega^2 = I_{id} \,. \tag{2.18}$$

This identification requires that the number of D0-branes should be equal to the number of $\overline{\text{D0}}$-branes. If one relaxes this condition, one may also considers the building of field configurations where the numbers of D0-branes and $\overline{\text{D0}}$-branes are not necessary equal. This violates however the $\mathbb{Z}_2$ symmetry

$$D_i = A_i^+ , \qquad H_i = B_i^+ \,, \tag{2.19}$$

and so will be omitted.

Irrational representations. The generalization of the previous case to irrational θ_i's is not automatic first because it uses more involved functional analysis and second its interpretation in terms of systems of D0-$\overline{\text{D0}}$ branes bound states. Following the same lines as for the rational case by working in a representation in which U_i is diagonal and using results on the non-commutative irrational torus, one can take the following realization

$$\begin{aligned}
\langle x'_{2i-1}|U_i|x_{2i-1}\rangle &= \begin{bmatrix} \mathrm{e}^{ix_{2i-1}} & 0 \\ 0 & \mathrm{e}^{-ix_{2i-1}} \end{bmatrix} \delta\left(x_{2i-1} - x'_{2i-1}\right) , \\
\langle x'_{2i-1}|V_i|x_{2i-1}\rangle &= \begin{bmatrix} \delta\left(x_{2i-1} + \theta_i - x'_{2i-1}\right) & 0 \\ 0 & \delta\left(x_{2i-1} - \theta_i - x'_{2i-1}\right) \end{bmatrix} ,
\end{aligned} \tag{2.20}$$

where we have set $R_I = 1/2\pi$ for commodity. The U_i and V_i operators depend on the x^{2i-1} variables and not on the x^{2i} ones. To construct the non-commutative orbifold projector operators on the position space generated by the continuous basis vectors $\{|x_{2i-1}\rangle \times |x_{2i}\rangle\}$, one may consider in a first attempt functions of the diagonal operator U_i. An apparently adequate candidate for the function $f(U_i)$ is given by:

$$\langle x'_{2i-1}|(\mathcal{P}_i)|x_{2i-1}\rangle = \langle x'_{2i-1}|f(U_i)|x_{2i-1}\rangle = \begin{pmatrix} \kappa_i & 0 \\ 0 & \kappa_i \end{pmatrix} \delta\left(x_{2i-1} - x'_{2i-1}\right) , \tag{2.21}$$

for $0 \leq x_{2i-1} \leq \kappa_i$, while for $\kappa_i < x_{2i-1} \leq 1$ we have

$$\langle x'_{2i-1}|f(U_i)|x_{2i-1}\rangle = 0 \,,$$

where κ_i is a priori a real parameter lying between zero and one. Though this choice of $(\mathcal{P}_i)$ ensures that it is hermitian, satisfies $(\mathcal{P}_i)^2 = (\mathcal{P}_i)$, it fails however as the trace $\mathrm{Tr}(\mathcal{P}_i)$ is not an integer in general since,

$$\mathrm{Tr}(\mathcal{P}_i) = \int \mathrm{d}x_{2i-1} \langle x_{2i-1} | (\mathcal{P}_i) | x_{2i-1} \rangle = 2\kappa_i \,. \tag{2.22}$$

This trace is not acceptable, it contradicts the expected spectrum dictated by the group $\mathbf{K}_0(\mathfrak{A}^o_{\theta_i}) = \mathbb{Z} + \theta_i \mathbb{Z}$, as κ_i is not quantized. To overcome this difficulty one should use both the U_i and V_i operators in building $\mathcal{P}_i$ instead of using U_i alone; this will allow to solve our problem and also incorporate explicitly the non-commutativity parameter into the game. Guided by the result on the non-commutative irrational torii, a class of solutions for the projector operators in agreement with $\mathbf{K}_0(\mathfrak{A}^o_{\theta_i})$ reads as:

$$\mathcal{P}_{N_i+M_i\theta_i} = \begin{pmatrix} P_{n_i+m_i\theta_i} & 0 \\ 0 & \overline{P}_{\overline{n}_1+\overline{m}_1\theta_i} \end{pmatrix}, \tag{2.23}$$

where N_i and M_i stand respectively for the multi-indices (n_i, m_i) and $(\overline{n_i}, \overline{m_i})$ and where

$$\begin{aligned} P_{n_i+m_i\theta_i} &= \left(\overline{B}_i^{m_i}\right)\left(g\left(\overline{A}_i\right)\right) + f\left(A_i\right) + g\left(A_i\right) B_i^{m_i}\,; \\ \overline{P}_{\overline{n}_i+\overline{m}_i\theta_i} &= \overline{g}\left(A_i\right) B_i^{\overline{m_i}} + \overline{f}\left(\overline{A}_i\right) + \left(\overline{B}_i^{\overline{m_i}}\right)\left(\overline{g}\left(\overline{A}_i\right)\right). \end{aligned} \tag{2.24}$$

The eigenvalues of the functions $f(A_i), \overline{f}\left(\overline{A}_i\right)$ and $g\left(A_i\right)$ and $\left(\overline{g}\left(\overline{A}_i\right)\right)$ are given by:

$$f\left(A_i\right) = \begin{cases} \dfrac{x^{2i-1}}{\epsilon_i} & x^{2i-1} \in [0, \epsilon_i] \\ 1 & x^{2i-1} \in [\epsilon_i, \theta_i] \\ 1 - \dfrac{x^{2i-1} - (n_i + m_i\theta_i)}{\epsilon_i} & x^{2i-1} \in [\theta_i, \theta_i + \epsilon_i] \\ 0 & x^{2i-1} \in [\theta_i + \epsilon_i, 1] \end{cases} \tag{2.25}$$

$$\overline{f}\left(\overline{A}_i\right) = \begin{cases} -\dfrac{x^{2i-1}}{\epsilon_i} & x^{2i-1} \in [-\epsilon_i, 0] \\ 1 & x^{2i-1} \in [-\theta_i, -\epsilon_i] \\ 1 + \dfrac{x^{2i-1} + (\overline{n}_i + \overline{m}_i\theta_i)}{\epsilon_i} & x^{2i-1} \in [-\theta_i - \epsilon_i, -\theta_i] \\ 0 & x^{2i-1} \in [-1, -\theta_i - \epsilon_i] \end{cases} \tag{2.26}$$

and

$$g\left(A_i\right) = \begin{cases} \dfrac{x^{2i-1}\left(\epsilon_i - x^{2i-1}\right)^{1/2}}{\epsilon_i} & x^{2i-1} \in [0, \epsilon_i] \\ 0 & x^{2i-1} \in [\epsilon_i, 1] \end{cases}, \tag{2.27}$$

$$\left(\overline{g}\left(\overline{A}_i\right)\right) = \begin{cases} \dfrac{-x^{2i-1}\left(\epsilon_i + x^{2i-1}\right)^{1/2}}{\epsilon_i} & x^{2i-1} \in [-\epsilon_i, 0] \\ 0 & x^{2i-1} \in [-1, -\epsilon_i] \end{cases}, \tag{2.28}$$

where ϵ_i is a small parameter which physically may be interpreted as a regulator parameter. Actually, the above result extends the constructions considered in [7] for the Powers-Rieffel

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projectors and in [10] for quantum torii. Observe here that the integers (n,m) and $(\overline{n},\overline{m})$ carried by the P and $\overline{P}$ projectors are not necessary equal as the constraint eqs. (2.11) are not violated. This means that in the irrational realization of the $\mathbb{Z}_2$ orbifold eq. (2.18) is no longer a necessary constraint contrary to the rational case. So for irrational $\mathbb{Z}_2$ orbifolds, one will be dealing with richer systems of D2-$\overline{\text{D}}$ branes as we shall see later on.

Having given the representations of $\mathfrak{A}^o_{\theta_i}$ for $\mathbb{T}^2_\theta/\mathbb{Z}_2$, we turn now to extend them to $\mathbb{T}^{2l}_\theta/\mathbb{Z}_2$. For fixed l, we have generally 2^l possibilities depending on whether the θ_i's are rational or irrational. If all θ_i's are rational, i.e. $\theta_i = q_i/p_i$ the U_i and V_i are given by similar eqs. to eq. (2.12). If instead all θ_i's are irrational, the U_i's and V_i's are given by

$$\langle \mathbf{x}'|U_i|\mathbf{x}\rangle = \begin{bmatrix} e^{ix_{2i-1}}\delta\left(\mathbf{x}-\mathbf{x}'\right) & 0 \\ 0 & e^{-ix_{2i-1}}\delta\left(\mathbf{x}-\mathbf{x}'\right) \end{bmatrix}, \tag{2.29}$$

$$\langle \mathbf{x}'|V_i|\mathbf{x}\rangle = \begin{bmatrix} \delta^{2l}\left(\mathbf{x}+\theta_i-\mathbf{x}'\right) & 0 \\ 0 & \delta^{2l}\left(\mathbf{x}-\theta_i-\mathbf{x}'\right) \end{bmatrix}. \tag{2.30}$$

We can also have the case where part of the θ_i's are rational and the others are irrational. In this case the U_i' s and V_i's are given by mixing the representations (2.12) and (2.20).

The projectors $\mathcal{P}_{\{\theta_1,\dots,\theta_l\}}$ on the position basis $\{|\mathbf{x}\rangle = |(x_1,x_2,\dots,x_{2l-1},x_{2l})\rangle\}$ for $\mathbb{T}^{2l}_\theta/\mathbb{Z}_2$ have then several forms depending on whether the θ_i's are rational or irrational. Denoting by $\mathcal{P}_{\{\theta_i\}}$ the projector operator associated to θ_i which is given by either eq. (2.12) or (2.23), we have

$$\mathcal{P}_{\{\theta_1,\dots,\theta_l\}} = \bigotimes_{i=1}^{l} \mathcal{P}_{\{\theta_i\}}\,. \tag{2.31}$$

From eq. (2.31), one learns that there are a priori 2^l solutions. However if one identifies operators that are related under permutations of positions, one ends then with l different objects. Note that the trace of eq. (2.31) is given by the product of the traces of the individual projectors $\mathcal{P}_{\{\theta_i\}}$; i.e.

$$\operatorname{Tr}\mathcal{P}_{\{\theta_1,\dots,\theta_l\}} = \prod_{i=1}^{l}\operatorname{Tr}\mathcal{P}_{\{\theta_i\}} = \prod_{i=1}^{l}\left(\operatorname{Tr}P_{\{\theta_i\}} + \operatorname{Tr}\overline{P}_{\{\theta_i\}}\right). \tag{2.32}$$

3. Non-BPS branes on orbifolds

Consider a non-BPS D2l-brane in presence of a $B_{\mu\nu}$ field on the NC orbifold $\mathcal{O} = \mathbb{T}^{2l}_\theta/\mathbb{Z}_2$ we introduced earlier and study the field configurations minimizing the total energy $E(T)$ of the tachyon living on the D2l-brane world volume. Keeping only the tachyon field $T(x)$ and integrating out all other fields, the string field theory effective action $\mathcal{S} = \mathcal{S}(T(x))$ reads as

$$\mathcal{S} = \frac{C_{\mathrm{D}2l}}{G_S}\int_{\mathcal{O}} d^{2l}x\sqrt{G}\left(\frac{1}{2}f\left(*T\right)G^{\mu\nu}\partial_\mu T\partial_\nu T + \cdots + V\left(*T\right)\right), \tag{3.1}$$

where G_S is the open string coupling constant and $G_{\mu\nu}$, $C_{\mathrm{D}2l}$ and the factor $f(t)$ are related by

$$
\begin{aligned}
G_{\mu\nu} &= g_{\mu\nu} - \left(2\pi\alpha'\right)^2 \left(Bg^{-1}B\right)_{\mu\nu} , && (3.2)\\
C_{\mathrm{D}2l} &= G_S M_{\mathrm{D}2l} . && (3.3)
\end{aligned}
$$

$G_{\mu\nu}$ and $g_{\mu\nu}$ being the effective open string and closed string metrics respectively and $f(T)$ is the effective coupling normalized like $f(0) = 0$ and $f(t_{\max}) = 1$ as suggested by Sen's conjecture.

In large non-commutativity, the kinetic term of the tachyon is neglected so that the action (3.1) reduces to $\mathcal{S} = \frac{C_{\mathrm{D}2q}}{G_S} \int_{\mathcal{O}} d^{2q}x\sqrt{G}V\,(*T)$. Therefore, the total energy $E(T)$ reads, upon taking $G_{\mu\nu} = \delta_{\mu\nu}$, as

$$
E(T) = M_{\mathrm{D}2l}\,\mathrm{Tr}\,V(T)\,, \tag{3.4}
$$

where $M_{\mathrm{D}2l}$ denotes the mass of the original D2l-brane and the trace Tr is normalized as $\mathrm{Tr}\mathbf{1} = 1$. Extremisation of $E(T)$ is achieved as usual by using the GMS approach which shows that the tachyon field configuration is proportional to the projectors in the $\mathfrak{A}_\theta$ NC algebra. The idea is based on taking the tachyon field $T(x)$ as

$$
T(x) = \sum_r t_r \mathfrak{p}_r(x)\,, \tag{3.5}
$$

where $\mathfrak{p}_r$ are mutually orthogonal projectors of $\mathcal{A}_\theta$ standing for Π_k eq. (2.15) or $\mathcal{P}_{n+m\theta}$ eq. (2.23) depending on whether θ is rational or irrational. Then using the GMS method, the total energy of the vacuum configurations can be shown to be proportional to the trace of the projectors as shown here below

$$
E = M_{\mathrm{D}2l} \sum_i V(t_i)\,\mathrm{Tr}(\mathfrak{p}_i)\,, \tag{3.6}
$$

the t_i's are the critical values solving $\frac{dV(t)}{dt} = 0$. Following the Sen's conjecture the tachyon potential $V(t)$ has two extrema; one minimum at the origin $t_{\min} = 0$ with $V(t_{\min}) = 0$ and a maximum at $t_{\max}$ with $V(t_{\max}) = 1$ and so

$$
T(x) = t_{\max}\mathfrak{p}(x)\,. \tag{3.7}
$$

Recall in passing that $t_{\max}$ and $t_{\min}$ describe respectively an unstable local maximum representing the space filling D2-brane ($V(t_{\max}) = 1$), and a local minimum representing the closed string vacuum without any D-branes ($V(t_{\min}) = 0$). The orbifold vacuum field configurations (3.6) lie between the original D2l-brane associated with $T = t_{\max}\mathbf{1}$ $= t_{\max}\sum_{r=1}^{\infty}\mathfrak{p}_r(x)$, and the complete tachyon condensation ($T = 0$). Let us discuss these configurations.

3.1 Branes on rational orbifolds

To start consider the case of a D2-brane on the NC orbifold $\mathbb{T}^2_\theta/\mathbb{Z}_2$ described by the non-commutative coordinate operators (A, B) and their conjugates eqs. (2.12). Level k solitons on the rational orbifold $\mathbb{T}^2_\theta/\mathbb{Z}_2$ are given by

$$T_k(x) = t_{\max}(\Pi)_k = \frac{t_{\max}}{p}\sum_{n=0}^{p-1}\frac{1-\omega^{-nk}}{1-\omega^{-n}}\begin{pmatrix}(A)^n & 0\\ 0 & (\overline{A})^n\end{pmatrix}. \tag{3.8}$$

Therefore the energy (3.6) reads as

$$E = \frac{t_{\max}}{2p}M_{\mathrm{D}q}\left(k+\overline{k}\right). \tag{3.9}$$

where $k = \overline{k}$. Following the analysis of [7] made for the quantum two torus and using general results on orbifold symmetries, in particular in compactification of Matrix model on orbifolds [11], the above equation can be interpreted in a nice way in terms of D0-$\overline{\text{D0}}$ branes systems. At the fixed points of the orbifold eq. (3.9) describes D0 branes but also anti-D0 branes as a consequence of the $\mathbb{Z}_2$ orbifold symmetry. More precisely it describes k coincident D0-branes and k coincident $\overline{\text{D0}}$-branes, which at low energies, are described by a non perturbative effective field theory with a $\mathrm{U}(k)\times\mathrm{U}(k)$ gauge symmetry. This result is automatically extendable to the higher dimensional orbifolds $\mathbb{T}^{2l}_\theta/\mathbb{Z}_2$. Thus starting from an unstable $D2l$ brane on $\mathbb{T}^{2l}_\theta/\mathbb{Z}_2$ and following the same lines we described earlier for the leading $l = 1$ case, one gets:

$$E_l = \sum_{i=1}^{l}\frac{t_{\max}}{2p_i}M_{\mathrm{D}2l}\left(k_i+\overline{k}_i\right). \tag{3.10}$$

This brane configuration has $\sum_{i=1}^{l} k_i$ number of D0-branes and the same number of anti D0-branes so that the gauge group of the underlying effective field theory is: $\mathrm{U}(\sum_{i=1}^{l} k_i)\times \mathrm{U}(\sum_{i=1}^{l} k_i)$. Finally we should note that a part the doubling of the spectrum, the analysis of tachyon condensation on NC orbifolds is quite similar to the condensation on rational torii. For details see [10].

3.2 Branes on irrational orbifolds

Non-BPS D2-branes wrapped on the irrational orbifold $\mathbb{T}^2_\theta/\mathbb{Z}_2$ lead to solitons proportional to the projectors on the ground states of the Hilbert space of the non-commutative algebra $\mathfrak{A}^o_\theta$. The $\mathfrak{p}_r$ projectors, which are given by an appropriate generalization of the Power-Rieffels operators, reads now as:

$$T_{N+M\theta} = t_{\max}\mathcal{P}_{N+M\theta} = t_{\max}\begin{pmatrix}P_{n+m\theta} & 0\\ 0 & \overline{P}_{\overline{n}+\overline{m}\theta}\end{pmatrix}. \tag{3.11}$$

The total energy of this field configuration is then

$$E = t_{\max}M_{\mathrm{D}2}[(n+\overline{n}) + (m+\overline{m})\theta], \tag{3.12}$$

while its index $\mathrm{Ind}(T)$ is

$$\mathrm{Ind}(T) = t_{\max} M_{\mathrm{D2}}[(n-\overline{n}) + (m-\overline{m})\theta]\,, \tag{3.13}$$

Moreover, setting $t_{\max} = 1$ and using the interpretation in terms of branes and anti branes as well as the following mass spectrum relations

$$M_{\mathrm{D2}} = M_{\overline{\mathrm{D}}2} = \sqrt{2}\frac{R_1 R_2}{g_s\,(\alpha')^{3/2}}\left(\left[1+\left(2\pi\alpha' B\right)^2\right]^{\frac{1}{2}}\right), \tag{3.14}$$

$$M_{\mathrm{D0}} = M_{\overline{\mathrm{D}}0} = \sqrt{2}\frac{1}{g_s\,(\alpha')^{1/2}} = \theta M_{\mathrm{D2}} = \theta M_{\overline{\mathrm{D}}2}\,. \tag{3.15}$$

where M_{D2} $(= M_{\overline{\mathrm{D}}2})$ and M_{D0} $(= M_{\overline{\mathrm{D}}0})$ are respectively the masses of the non BPS D2 $(\overline{\mathrm{D}}2)$ and of D0 ($\overline{\mathrm{D}}0$) branes and where $B = 1/2\pi R_1 R_2 \theta$, the energy of the vacuum configuration can be split as

$$E = (nM_{\mathrm{D2}} + \overline{n}M_{\overline{\mathrm{D}}2}) + \theta(mM_{\mathrm{D0}} + \overline{m}M_{\overline{\mathrm{D}}0}), \tag{3.16}$$

This formula means that the original M_{D2} annihilates to D0-D2, D0- $\overline{\mathrm{D}}2$, $\overline{\mathrm{D}}0$-D2 and $\overline{\mathrm{D}}0$-$\overline{\mathrm{D}}2$ bound states. This interpretation is not new as it has been first given in [7] in the context of the study of D2-brane on the irrational two torus annihilating to D0-D2 bound states. It is dictated by the analysis on tachyon condensation and also supported by T-duality as well as the exact mass spectrum of the $\{mD0,\ \overline{m}\overline{\mathrm{D}}0,\ nD2,\ \overline{n}\overline{\mathrm{D}}2\}$ bound states

$$M_{(n,m)} = M_{(\overline{n},\overline{m})} = \sqrt{2}\frac{R_1 R_2}{g_s\,(\alpha')^{3/2}}\left[1+\left(2\pi\alpha' B_{\mathrm{eff}}\right)^2\right]^{\frac{1}{2}}\,, \tag{3.17}$$

where the effective B_{eff} field is

$$B_{\mathrm{eff}} = B + \frac{1}{2\pi R_1 R_2}\frac{m}{n}\,. \tag{3.18}$$

Taking the large limit of $(2\pi\alpha' B_{\mathrm{eff}})$, eq. (3.16) is then reproduced. The novelty here is that one is in presence of a richer spectrum of bound states involving branes and anti branes. The remaining analysis is like in the quantum two torus case. In what follows, let us give some general results of branes on NC orbifolds.

3.3 Fractional branes

Recall that whenever we have commutative orbifolds with discrete torsion, there exist fractional branes at singularities [16]. The point is that for the rational representation of $\mathbb{T}^2/\mathbb{Z}_2$ we will be discussing here below, one should distinguish between bulk (regular) and singular representations. Though it seems a little bit ambiguous to talk about singularity in NC geometries, we will see that there is indeed situations where NC geometry and singular representations may go together. In presence of $\mathbb{Z}_2$ discrete torsion carried by the algebra

$$\mathbf{pq} = \alpha\mathbf{qp}\,, \qquad \alpha^2 = 1 \tag{3.19}$$

with

$$\mathbf{p} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.20}$$

the coordinate variables x^1 and x^2 becomes non commutative 2×2 matrix opertaors X^1 and X^2 even in absence of the antisymmetric NS-NS $B_{\mu\nu}$ field. Indeed in the case where $B_{\mu\nu} = 0$, these X's obey, in addition to the commutation relation $X^1X^2 = X^2X^1$, the constraint equation $\mathbf{p}X^i\mathbf{p}^{-1} = \alpha X^i$ defining the $\mathbf{Z}_2$ orbifold. Putting together these realtions, one sees that the X's generate the following NC algebra $\mathcal{A}_\alpha$

$$\begin{aligned} X^1X^2 &= X^2X^1\,, \\ \mathbf{p}X^i &= \alpha X^i\mathbf{p}\,, \qquad \alpha^2 = 1\,. \end{aligned} \tag{3.21}$$

In presence of torsion, that is for $\alpha = -1$, the above relations may be solved by using the $\mathbf{p}$ and $\mathbf{q}$ operators of eq. (3.20) as shown herebelow,

$$\begin{aligned} X^1 &= y^1 \otimes \mathbf{q}\,, \\ X^2 &= y^2 \otimes \mathbf{q}^{-1}\,, \\ \mathbf{p}^2 &= \mathbf{I}_2\,, \end{aligned} \tag{3.22}$$

where y^1 and y^2 are real moduli. This solution which is singular at the origin $y^1 = y^2 = 0$; describes a non commutative torus $\mathbb{T}^2_\alpha/\mathbb{Z}_2$ where now the NC geometry is generated by discrete torsion. The original A_1 singularity appearing in the case $\alpha = 1$, is resolved by NC geometry. To see this feature, let us introduce the $\mathbb{Z}_2$ invariants $u = (X^1)^2$, $v = (X^2)^2$ and $z = X^1X^2$ satisfying the familiar equation

$$uv = z^2\,. \tag{3.23}$$

In the special case where the torsion is absent, $\alpha = 1$, the $\mathbf{p}$ and $\mathbf{q}$ operators become trivial ($\mathbf{p} = \mathbf{q} = \mathbf{1}$) and the geometry is singular at $X^1 = y^1 = 0$ and $X^2 = y^2 = 0$. In presence of torsion ($\alpha = -1$), the operators $\mathbf{p}$ and $\mathbf{q}$ are given by eqs. (3.20) and so the point like $y^1 = y^2 = 0$ singularity is replaced by a non trivial fiber; a non zero loop base on $y^1 = y^2 = 0$. At singularity, the $X^1 = X^2 = 0$ and one is left with the reducible $\mathbf{p}$ operator showing that we have two fractional branes at this point.

In presence of both the antisymmetric NS-NS $B_{\mu\nu}$ field and the $\mathbb{Z}_2$ discrete torsion, the y^1 and y^2 variables become operators Y^1 and Y^2 playing the same role as the X's of eqs. (2.3) and (2.4). The resulting NC geometry $\mathbb{T}^2_{\alpha,\theta}/\mathbb{Z}_2$ has now two deformation parameters α and θ, one coming from discrete torsion and the other due to the presence of the NS-NS B fields. In this case eq. (3.22) become

$$\begin{aligned} X^1 &= Y^1 \otimes \mathbf{q}\,, \\ X^2 &= Y^2 \otimes \mathbf{q}^{-1}\,, \\ \mathbf{p}^2 &= \mathbf{I}_2\,. \end{aligned} \tag{3.24}$$

while the new variable operators of the NC torus $\mathbb{T}^2_{\alpha,\theta}/\mathbb{Z}_2$ with discrete torsion are given by

$$\begin{aligned} \widehat{U} &= e^{\frac{i2\pi}{R_1}Y^1\otimes\mathbf{q}} \\ \widehat{V} &= e^{\frac{i2\pi}{R_2}Y^2\otimes\mathbf{q}^{-1}}\,, \\ \mathbf{p} &= \mathrm{diag}\,(1,-1)\,. \end{aligned} \tag{3.25}$$

Note that eqs. (3.24) are no longer singular as far as the NS-NS $B_{\mu\nu}$ field is non zero since in this case the Y variables cannot be zero due to the constraint eq. $\left[Y^1, Y^2\right] = \theta \sim 1/B$. However for the special limit where θ goes to zero, but there is still a non trivial torsion cocyle; that is in the case where the NC torus $\mathbb{T}^2_{\alpha,\theta}/\mathbb{Z}_2$ becomes $\mathbb{T}^2_{\alpha}/\mathbb{Z}_2$, the y^1 and y^2 variables can take the zero value and so eqs. (3.25) should be read as

$$\begin{aligned} \widehat{U} &= \lim_{y^1 \to 0} e^{\frac{i2\pi}{R_1} y^1 \otimes \mathbf{q}}, \\ \widehat{V} &= \lim_{y^2 \to 0} e^{\frac{i2\pi}{R_2} y^2 \otimes \mathbf{q}^{-1}}, \\ \mathbf{p} &= \mathrm{diag}\,(1, -1)\,. \end{aligned} \tag{3.26}$$

Therefore the representation becomes completely reducible at singularity and so there are two fractional branes associated with to the eigenvalues of $\mathbf{p}$. Such analysis can be extended to higher dimensional torii without difficulty. For the non-commutative $2l$ dimensional torii $\mathbb{T}^{2l}_{\alpha,\Theta}/Z_2$ with discrete torsion given by the product of l NC two torii $\mathbb{T}^2_{\alpha_i,\theta i}/Z_2$; i.e. $\mathbb{T}^{2l}_{\alpha,\Theta}/Z_2 = \otimes_{i=1}^{l} \mathbb{T}^2_{\alpha_i,\theta i}/Z_2$ generated by the coordinate system

$$\begin{aligned} X^{2i-1} &= Y^{2i-1} \otimes \mathbf{q}_i\,, \\ X^{2i} &= Y^{2i} \otimes \mathbf{q}_i^{-1}\,; \qquad 1 \le i \le l\,, \\ \left[Y^{2i-1}, Y^{2i}\right] &= \theta_i\,, \end{aligned} \tag{3.27}$$

the l unitary operator pairs $\left(\widehat{U}_i, \widehat{V}_i\right)$ reads as

$$\begin{aligned} \widehat{U}_i &= e^{\frac{i2\pi}{R_1} Y^{2i-1} \otimes \mathbf{q}}, \\ \widehat{V}_i &= e^{\frac{i2\pi}{R_2} Y^{2i} \otimes \mathbf{q}^{-1}}, \\ \mathbf{p}_i \mathbf{q}_i &= \alpha_i \mathbf{q}_i \mathbf{p}_i\,, \qquad \alpha_i^2 = 1 \end{aligned} \tag{3.28}$$

where $\mathbf{p}_i = \mathrm{diag}(1, -1)$ and $\mathbf{q}_i$ as in eq. (3.20). Here also we have a multi-parameter $(\alpha_1, \ldots, \alpha_l; \theta_1, \ldots, \theta_l)$ NC geometry and as one sees, according to the values of θ_i and α_i, one may distinguish several cases of NC $\mathbb{T}^{2l}_{\alpha,\Theta}/Z_2$. Since fractional branes exist whenerver torsions are taken into account ($\alpha_i = -1$) and as far as the NS-NS B fields is very large ($Y^{2i-1}Y^{2i} = Y^{2i}Y^{2i-1}$), we find fractional branes at all singular points of $\mathbb{T}^2_{\alpha_i,\theta i}/Z_2$ with $\alpha_i = -1$ and $\theta_i = 0$.

4. More on bound states

So far, we have discussed vacuum energy configurations of a tachyon field on orbifolds for the two representations: rational and irrational. In the former, we have learned that the energy of the D and $\overline{\mathrm{D}}$ brane states is bounded by the initial D-brane mass M_{D} and we have interpreted this feature by saying that the number of D and $\overline{\mathrm{D}}$-branes filling in the orbifold is limited; the number of D or anti D-brane can not exceed p, the weight of the rational representation of the orbifold. In the irrational case, the total energy is still bounded by M_{D}, however due to the continuous property of states density, one is left with an unusual system of D and anti D brane bound states. In the case of a D2-brane on a NC irrational two torus, the bound states one has are of type mD0-nD2; they were interpreted as describing k D0-branes where k is the greatest common divisor of the n and m integers. This idea has been extended in [10] to the case of a D-branes wrapped on higher dimensional torii. There we have found general bound states involving $\{$D2s $0 \leq s \leq l\}$ brane systems. In the case of D2-brane on the $\mathbb{T}^2_\theta/\mathbb{Z}_2$ orbifold, the situation is quite similar, the main difference with $\mathbb{T}^2_\theta$ is the presence of extra bound states involving anti D branes as shown in eq. (3.16). This result extends naturally to higher dimensions. To determine the gauge symmetry of the underlying field theory, we proceed as for the NC torus since the system still inherits the torus T-duality symmetry. In deed, since the total energy spectrum bound states configurations

$$E\left(\mathcal{P}_{N+M\theta}\right) = \left[(n+\overline{n}) + (m+\overline{m})\theta\right]\frac{V}{G_s} \tag{4.1}$$

is invariant under SL$(2,\mathbb{Z})$, i.e.

$$\begin{pmatrix}\alpha' \\ \beta'\end{pmatrix} = \begin{pmatrix} s & -r \\ -q & p\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}.$$

where $sp - rq = 1$, and where α, β stand for n, m, $\overline{n}$ and $\overline{m}$. one can usually set $k = n/r$ and $\overline{k} = \overline{n}/r$. Therefore the gauge symmetry is U(k) $\times$ U$(\overline{k})$ with k and $\overline{k}$ the greatest common divisor of (n, m) and $(\overline{n}, \overline{m})$ respectively. In other words, the system is similar to parallel k D0-brane and $\overline{k}$ $\overline{\text{D}}$0-branes.

5. D-$\bar{\text{D}}$ systems

In this section we would like to give some comments regarding brane anti brane condensations and also compare our results on tachyons condensation on orbifolds with the analysis of [7] dealing with D2-$\overline{\text{D}}$ systems on NC torii There, it has been shown that starting from a pair of D2-brane anti-D2-brane system on the two non commutative torus, the total energy may be written in the large B limit as:

$$E(T,\overline{T}) = M_{\text{D2}}Tr[V(1 - T\overline{T}) + V(1 - \overline{T}T)]\,, \tag{5.1}$$

where T and $\overline{T}$ are the complex tachyon and its conjugates. These tachyon fields are treated a little bit differently from the usual real tachyon field; they are interpreted as as linear maps interpolating between the two Hilbert spaces $\mathcal{H}$ and $\overline{\mathcal{H}}$

$$T : \mathcal{H} \to \overline{\mathcal{H}}\,, \qquad \overline{T} : \overline{\mathcal{H}} \to \mathcal{H}$$

and satisfying the following eqs. which may roughly be thought of as a complexification of the GMS anzats

$$T = T\overline{T}T\,, \qquad \overline{T} = \overline{T}T\overline{T}\,. \tag{5.2}$$

Using these relations, projectors Π and $\overline{\Pi}$ on the vacuum subspaces of $\mathcal{H}$ and $\overline{\mathcal{H}}$ have been built in [8] and can be shown to be given by

$$\Pi = 1 - T\overline{T}\,, \tag{5.3}$$

$$\overline{\Pi} = 1 - \overline{T}T\,. \tag{5.4}$$

These are self adjoint operators verifying $\Pi^2 = \Pi$ and $\overline{\Pi}^2 = \overline{\Pi}$ together with other relations specifying their actions on T and $\overline{T}$. Using the modified Power-Rieffels realization of Π and $\overline{\Pi}$ on the non commutative torus, see eqs. (34) of [7], as well as the above identities (5.3), the total energy formula (5.1) becomes then

$$\begin{aligned} E(T,\overline{T}) &= M_{\mathrm{D2}}\mathrm{Tr}[\Pi + \overline{\Pi}] \\ &= M_{\mathrm{D2}}[(n+m)\theta] + M_{\overline{\mathrm{D2}}}[\overline{n} + \overline{m}\theta]\,. \end{aligned} \tag{5.5}$$

Similar relations such as for the index $\mathrm{Ind}(T,\overline{T})$ may be also written down. We shall not give them explicitly here, they may be obtained immediately from the analysis of section 3 and the following discussion. What we want to make now is to give some comments about this system. The first thing one may note about the above construction concerns the potential of the complex tachyon. This is a striking U(1) invariant function from which one already learns the essential about the vacuum configuration projectors and so about the condensation of the D-$\overline{\mathrm{D}}$ systems. This construction is a priori extendable to D-$\overline{\mathrm{D}}$ systems involving various numbers of original brane and anti branes and seems to lead to more general GMS type relations extending eqs. (5.2) and (5.3). For example if instead of the complex T and $\overline{T}$ fields, one considers the following 2×2 matrix valued fields T and $\overline{T}$

$$\begin{aligned} T &= T_0 I + \sum_{i=1}^{3} T_i \sigma^i\,, \\ \overline{T} &= \overline{T}_0 I + \sum_{i=1}^{3} \overline{T}_i \sigma^i\,, \end{aligned} \tag{5.6}$$

where (T_0, T_i) and $(\overline{T}_0, \overline{T}_i)$ are four complex tachyon fields and their complex conjugates and where the σ^i's are the usual Pauli matrices, the analogue of eqs. (5.2) and (5.3) reads as

$$\begin{aligned} T &= TT^\dagger T\,, \qquad & T^\dagger &= T^\dagger T T^\dagger\,, \\ \Pi &= 1 - TT^\dagger\,, \qquad & \overline{\Pi} &= 1 - T^\dagger T\,. \end{aligned} \tag{5.7}$$

In this case the energy and the index of the vacuum configuration is invariant under a U(2) automorphism group rotating the different tachyons. Nevertheless if one is not insisting on automorphism symmetries, in particular on the above mentioned U(1) symmetry of [7] and considers instead real tachyons T_1 and T_2 interchanged under a $\mathbb{Z}_2$ symmetry, one gets the same spectrum of brane states in the condensation. Indeed the total energy and the index of the real D2 brane-anti D2 brane configuration on the non-commutative two torus reads in large non-commutativity

$$\begin{aligned} E &= E(T_1) + E(T_2) = M_{\mathrm{D2}} TrV(T_1) + M_{\overline{\mathrm{D2}}} TrV(T_2)\,, \\ I_{\mathrm{nd}} &:= \mathrm{Index}(T_1, T_2) = M_{\mathrm{D2}} TrV(T_1) - M_{\overline{\mathrm{D2}}} TrV(T_2)\,. \end{aligned} \tag{5.8}$$

Since in this limit T_1 and T_2 are not coupled, they obey then exactly to the standard rule giving the GMS solitons on the quantum two torus. A direct check shows that after straightforward calculations, one ends with the same states spectrum as in the previous complex approach and so the same conclusion. However and as far as concluding about the D-$\overline{\text{D}}$ system is concerned, we still have something else to add. This concerns the way the $\mathbb{Z}_2$ symmetry involved in the game. Besides the two ways we discussed above where $\mathbb{Z}_2$ acts directly on the tachyon fields but not on the noncommutative manifold on which the brane and anti brane are wrapped, one can also consider the general case where $\mathbb{Z}_2$ does act on the quantum manifold. This is the case of $\mathbb{Z}_2$-orbifolds we have been discussing in this paper. Let us briefly summarize the thing.

6. Summary and discussion

Z_2 orbifolds and D-$\overline{\text{D}}$ systems. Through our discussion, we have learned in the case of an original $D2$ brane wrapped on the non commutative orbifolds $\mathcal{O}_\theta$ that due the constraint eqs. (2.9), $\mathbb{Z}_2$ symmetry leads to a natural doubling of the torii Hilbert space spectrum interpreted as corresponding to brane and anti branes. The latest emerge automatically because of the orbifold Z_2 symmetry and do not need to introduce an original anti-D2 brane. For selfcontainess of this discussion let recall some of our results and compare to what we said in the beginning of this section. In the case of irrational orbifolds for instance, we have shown that the vacuum field configuration for the tachyon field is

$$T_{N+M\theta} = t_{\max} \begin{pmatrix} P_{n+m\theta} & 0 \\ 0 & \overline{P}_{\overline{n}+\overline{m}\theta} \end{pmatrix}, \tag{6.1}$$

where $P_{n+m\theta}$ and $\overline{P}_{\overline{n}+\overline{m}\theta}$ are generalization of Power Rieffels projectors given by eq. (2.24) and the energy E and the index I_{nd} of the soliton configurations are identical to (5.1). To see why these results match, we give hereafter an algebraic argument to explain the equivalence of the approaches.

The non-commutative orbifold $\mathcal{O}_\theta$ associated with the non commutative algebra $\mathcal{A}_\theta^o$(Orbifold) is much more constrained than the usual non-commutative algebra $\mathcal{A}_\theta$(Torus). Because of these constraints, representations $\mathcal{H}^o$(Orbifold) of $\mathcal{A}_\theta$(Orbifold) have twice the dimension of $\mathcal{H}$(Torus) $\equiv \mathcal{H}$; the $\mathcal{A}_\theta$(Torus) representations and then the $\mathbb{Z}_2$ symmetry is manifested through the following splitting .

$$\mathcal{H}^o = \mathcal{H} \oplus \overline{\mathcal{H}}\,, \tag{6.2}$$

where $\overline{\mathcal{H}}$ is the symmetric of $\mathcal{H}$. Note that that the above decomposition has in fact a larger automorphism group containing the orbifold discrete $\mathbb{Z}_2$ group just as a subsymmetry. To make an idea on this equivalence, it is enough to note that under the non commutative orbifold coordinate choice eqs. (2.9), one sees that $\mathcal{A}_\theta^o$(Orbifold) is simply

$$\mathcal{A}_\theta^o(\text{Orbifold}) = \mathcal{A}_\theta(\text{Torus}) \oplus \overline{\mathcal{A}_\theta}(\text{Torus}) \tag{6.3}$$

where $\overline{\mathcal{A}_\theta}$(Torus) is the image of $\mathcal{A}_\theta$(Torus) under $\mathbb{Z}_2$. So operators of $\mathcal{A}_\theta^o$(Orbifold) and too particularly the projectors split into two irreducible factors describing branes and anti branes.

Beyond the GMS model. Inspired from general properties of quaternions; we have proposed a way to extend the GMS construction for the tachyon potential and its extension on the quantum torus. In the above discussion we have considered the case of a system of D-$\overline{\mathrm{D}}$ branes transforming in a U(2) representation eqs. (5.2)–(5.4) and (5.7); let us consider hereafter the case of a system of D branes in U(2) adjoint and look for solutions minimising the tachyon potential. For this, let $T = \frac{1}{2}(t_0 I + t_i \sigma^i)$ be the real tachyon matrix field of the D brane system with σ^i the usual Pauli matrices satisfying $\sigma^i\sigma^j + \sigma^j\sigma^i = \delta^{ij}$ and t_0 and t_i are such that,

$$t_0 * t_0 = t_0 \,; \qquad t_i * t_i = t_i \,; \qquad t_0 * t_i = t_i \,, \qquad t_0 = \sum_i t_i \,. \tag{6.4}$$

The two first relations are of type GMS while the second ones are required by consistency. A way to think about these relations is to take t_0 as the sum of three projectors, say $t_0 \sim \Pi_{l_1} + \Pi_{l_2} + \Pi_{l_3}$ and $t_i \sim \Pi_{l_i}$ and $t_0 \sim \mathcal{P}_{l_1} + \mathcal{P}_{l_2} + \mathcal{P}_{l_3}$ and $t_i \sim \mathcal{P}_{l_i}$ for the case of the rational eq. (2.15) and irrational torii eq. (2.23) respectively. The projectors Π_{l_i} ($\mathcal{P}_{l_i}$) satisfy the usual identity $\Pi_{l_i}\Pi_{l_j} = \delta_{ij}\Pi_{l_i}$. With these relations at hand it is not difficult to check that the GMS method applies as well for tachyon matrix field T. Indeed computing $T * T$, we get

$$T * T = \frac{1}{4}\left(t_0 * t_0 + \sum t_i * t_i\right) I + \frac{1}{4}\sum \left(t_0 * t_i + t_i * t_0\right)\sigma^i \tag{6.5}$$

and using eqs. (6.4), we get indeed,

$$T * T = T \,. \tag{6.6}$$

One may also write down the effective action $\mathcal{S}$ for this tachyonic system for non zero NS antisymmetric magnetic field $B_{\mu\nu}$ by extending the analysis of [5]. In the case of the bosonic open string field theory, a D25 brane wrapped on a quantum orbifold torus, the effective action is,

$$\mathcal{S} = \frac{g_s}{G_s} T_{25} \int d^{26}x \sqrt{G} Tr \left(\frac{1}{2} f\left(T\right) G^{\mu\nu} \partial_\mu T \partial_\nu T + \cdots - V\left(T\right)\right) \tag{6.7}$$

where the star product is understood and where the matrix functions $f(T)$ and $V(T)$ are constrained by Sen's conjecture. The minimum of this action, which reduces to that of the potential in large non commuatitvity, is ensured by eq. (6.6). In the end, it is interesting to note that such analysis, which may be extended further by considering higher dimensional Dirac matrices, is not valid for U(n) groups with $n > 2$.

7. Conclusion

In this paper we have studied tachyon condensation in non commutative orbifolds. We have studied, amongst others, soliton solutions for these compact manifolds and derived in particular the generalization of the Power-Rieffels configurations on the irrational non commutative orbifolds with a Z_2 discrete symmetry. More precisely starting from an original non-BPS $D2l$ brane of mass $\mathrm{M}_{\mathrm{D}2l}$, $l = 1, 2, \ldots$ wrapped on $\mathbb{T}^{2l}/\mathbb{Z}_2$, we have studied

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its condensation using GMS formalism and related ideas and shown: (a) the existence of general bound states extending both those studied in [7] and [10] and containing amongst others $\overline{\mathrm{D}}$-$\overline{\mathrm{D}}$ bound states. (b) using general relations on the brane spectrum and T-duality, we have shown that the suggestion of Bars et al. regarding the interpretation of the D0-D2 branes, which by the way extends to more general D-D bounds, applies as well to $\overline{\mathrm{D}}0$-$\overline{\mathrm{D}}2$ bound states and more generally to $\overline{\mathrm{D}}$-$\overline{\mathrm{D}}$ bounds. (c) the condensation incorporates naturally D-$\overline{\mathrm{D}}$ brane systems on quantum torii without needing to introduce an original anti-D brane. We have demonstrated the equivalence of the D-$\overline{\mathrm{D}}$ brane system we have obtained, the $\mathbb{Z}_2$ symmetry of the orbifold do the full job. Finally we would like to note that we expect that our results extend as well to a large variety of toric orbifolds with other discrete symmetries.

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